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The Hopf fibration—seven times in physics

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Abstract

It is pointed out that the Hopf fibration—a special but very basic non-trivial principal fiber bundle—occurs in at least seven different situations in theoretical physics in various guises. Surprisingly, the gauge theory aspect is in the minority here.

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1. Introduction

The Hopf fibration as a purely mathematical idea has been around since 1931 when it allowed Hopf[1] to determine the third homotopy group of the 2-sphere and to show, in particular, that this group is non-trivial, by exhibiting a suitable map from the 3-sphere to the 2-sphere and the fibration of the 3-sphere related to it. It was realized only much later that this fibration occurred in the same year well-hidden in a physics context when Dirac extended the framework of wave mechanics by admitting for "wave functions" what we now call sections of non-trivial complex line bundles, and studied, in particular, quantum mechanical motion in the field of a magnetic monopole. The basic definition of the Hopf fibration is, however, also closely related to the distinction between state vectors and states in quantum mechanics, and its base space—the 2-sphere $\mathbf{S_2}$—appears literally in the case of two-level systems ("qubits", as they have been called recently)

[1] [23]; later Hopf invented more fibrations that now bear his name, but we will stick strictly to the 1931 construction in this article.

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as the so-called “Bloch sphere”, as was pointed out, e.g., by Penrose. Without recognition of the nature of the base space, a directly related fibration was—as remarked by Hopf—considered in the context of the parallelism invented by Clifford in 1873, couched, however, in the language of real projective 3-space (elliptic version of non-Euclidean geometry). A very direct occurrence, without any obvious role for the nature of the base space, appears when the phase space flow of the two-dimensional harmonic oscillator is restricted to a non-trivial energy shell. A more tricky occurrence of the fibration is in Penrose’s theory of twistors, where it serves to illustrate one special geometrical realization of a (projective) twistor. More features of the Hopf fibration are involved in its occurrence in General Relativity, namely in the global structure of Taub-NUT space: the nature of its base space and the linking of its fibers play an essential role in the study of that space. Complex line bundles associated with the Hopf fibration appear directly when the zero mass helicity representations of the (covering group of the) Poincaré group are considered (“Wignerism”), the non-trivial topological features making themselves felt at various instances, inhibiting the procedure of going from finite to zero mass. Finally, the Hopf fibration determines the spin structure of the 2-sphere, a concept entering not only “two-dimensional Wick-rotated physics”, but appearing also upon separation of the ordinary Dirac equation in spherically symmetric external electrostatic fields—although this fact is seldom mentioned in conventional treatments of the relativistic hydrogen spectrum.

This remarkable multiple occurrence of Hopf’s fiber bundle in at least seven different physical situations indicates that it is a basic geometrical element, possibly useful in the physical description of more situations, rather than a mathematical curiosity which most physicists did not know about before the mid-1970s. In the words of Penrose, it may be regarded as an “element of the architecture of our world”. Naturally then, the present article is more a matter of contemplation rather than research. In Appendix A, we will include the definition and a brief indication of several mathematical features of the fibration together with textbook references; the reader is advised to consult, if necessary, Appendix A parallel to the main text. In Section 2, the appearance of the fibration in two-level quantum systems is described, including a “connection” in the bundle that is related to adiabatic transport and Berry’s geometrical phase. This is, in a sense, a “top-down” version of the description, and in this form the fibration makes its appearance also in the two-dimensional isotropic harmonic oscillator (Section 3), in Taub-NUT space (Section 4) and in twistor theory (Section 5). Its is only in Wigner’s helicity representations of the Poincarë group (Section 6) and in Dirac’s monopole quantization (Section 7) that the bottom-up view—cherished by us physicists since we learn it this way in gauge theory—comes first; in fact, it is only the latter example where the scheme “local gauge potential, local field strength”, augmented by global considerations because of singularities in the local, gauge-dependent quantities, leads to the well-known patching construction of the fibration. We close by pointing out the role of the fibration in giving the 2-sphere’s spin structure (Section 8), a concept necessary for global considerations involving the Dirac equation and the Dirac operator.

Our collection could result in the moral that there are perhaps more fiber bundles in physics than a local, gauge theory shaped eye allows to see, and that, therefore, there may be some merit for us physicists in also looking at the non-differential geometric methods of studying them.

2. Hopf and the qubit—two-level quantum systems

The occurrence of the Hopf fibration in two-level quantum systems—systems that can be described using a two-dimensional complex Hilbert space, also nowadays known as *qubits*—was mentioned, e.g., by Penrose [36], and is described more extensively in [48]; so we can be brief and somewhat schematic about it here.

We take the two-dimensional Hilbert space just to be $\mathbf{C^2}$ with its standard Hermitian inner product $\langle z, w\rangle := \bar{z}_1 w_1 + \bar{z}_2 w_2 = z^\dagger w$. The states of the system are given by density matrices, i.e., Hermitian-positive 2×2 matrices of trace 1; the observables of the system are given by Hermitian matrices; the expectation values of the observable with matrix A in the state with density matrix ρ is given by the real number $\mathrm{Tr}\, A\rho$. When ρ has rank 1, it has the form of a one-dimensional projector, $\rho = zz^\dagger/z^\dagger z$, and is called a pure state, represented by the (non-zero) state vector $z \in \mathbf{C^2}$, and the expectation value of A becomes $z^\dagger Az/z^\dagger z$. One notes that a pure state determines the state vector only up to a non-zero complex factor, and even when z is required to be normalized, $\langle z, z\rangle \equiv z^\dagger z = 1$, a phase factor $\mathrm{e}^{\mathrm{i}\alpha}$, α real, remains undetermined. This of course means that the pure states can also be identified with the points of the projectivized space $\mathrm{P}(\mathbf{C^2}) = \mathbf{CP_1}$, but for the moment we will stick to the density matrices.

Consider now the expectation values of three observables $\sigma_1, \sigma_2, \sigma_3$, where

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

(These are known as the Pauli spin matrices, since they relate to the components of the spin angular momentum observables of a spin $\frac{1}{2}$ particle; but for our formal purposes here this interpretation is not essential.) Collecting them into a matrix vector $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$, we have for its expectation value in a state ρ the vector $\boldsymbol{R} := \mathrm{Tr}\, \rho\boldsymbol{\sigma} \in \mathbf{R^3}$. One can verify that this relation can be inverted as $\rho = \frac{1}{2}(1 + \boldsymbol{R}\cdot\boldsymbol{\sigma})$, where the positivity of ρ implies $\boldsymbol{R}^2 \leq 1$, equality holding precisely for pure states. This means that $\boldsymbol{R}(z) := z^\dagger\boldsymbol{\sigma}z/z^\dagger z$ satisfies $\boldsymbol{R}^2(z) = 1$. In other words, the density matrices ρ may, in the qubit case, be represented by vectors in $\mathbf{R^3}$ belonging to the closed unit ball $\mathbf{B_3}$ (the "Bloch sphere", in physics jargon, since it was used by F. Bloch to illustrate magnetic spin resonance phenomena), and the state is pure iff the representing $\mathbf{R^3}$-vector is a unit vector.

We can also get a geometric picture of the state vectors z themselves by looking at $\mathbf{C^2}$ as being $\mathbf{R^4}$, taking the real and imaginary parts of z_1, z_2 (in some order) as its real components. Then the assignment $z \mapsto \boldsymbol{R}(z)$ gives us a map $\mathbf{R^4} \to \mathbf{S_2} \subset \mathbf{R^3}$, and restricting to normalized z, $z^\dagger z = 1$, whose realifications fill the 3-sphere $\mathbf{S_3} \subset \mathbf{R^4}$, we get a map $\mathbf{S_3} \to \mathbf{S_2}$. This is the *Hopf map*: if the real components of z are numbered suitably and the definition of $\boldsymbol{R}(z)$ is written out explicitly in terms of them, the above expression for the latter becomes literally identical to Hopf's original formulae. It is easy to check that the inhomogeneous coordinate $\zeta = z_2/z_1$ on the space of pure states when looked at as $\mathbf{CP_1}$ is just the complex stereographic coordinate on the Bloch 2-sphere when this sphere is projected onto its equatorial plane from its south pole. The inverse images of the points on the Bloch 2-sphere under the Hopf map are "phase circles" on the 3-sphere. The 2-parameter system of phase circles on the 3-sphere so obtained constitute its *Hopf fibration*. In [48], the author explains the visualization of this system of circles by stereographic projection of the 3-sphere onto its equatorial 3-plane, where it appears as the system of "Villarceau circles" on a nested family of coaxial concentric tori orthogonal to the unit 2-sphere.

In [48], the author describes the idea of Penrose how to represent normalized state vectors z in terms of the Bloch sphere picture: one assigns to z not only the phase-insensitive position $\boldsymbol{R}(z)$, but also the phase-sensitive unit tangent vector at that position given by $\boldsymbol{P}(z) := \Re \boldsymbol{Z}(z)$, where $\boldsymbol{Z}(z) := \{z, \boldsymbol{\sigma} z\}$ with $\{z, w\} := z_1 w_2 - z_2 w_1$. When z undergoes a phase change, this tangent is just rotated through twice the phase angle.

In [48], the author further discussed adiabatic transport of state vectors over curves in the space of pure states, as specialized from the general case to the qubit case. For a given curve $\tau \mapsto \boldsymbol{R}_\tau$ on the 2-sphere, the transport $\tau \mapsto z_\tau$ is required to satisfy the differential equation $\langle z, (\mathrm{d}z/\mathrm{d}\tau)\rangle = 0$, i.e., z_τ is to be a curve on $\mathbf{S_3}$ having $\boldsymbol{R}(z_\tau) = \boldsymbol{R}_\tau$ whose tangent vectors are annihilated by the *connection form* $\langle z, \mathrm{d}z\rangle \equiv z^\dagger\,\mathrm{d}z$. It is shown there explicitly that the associated $\boldsymbol{P}(z)$ undergo Levi–Cività transport (which again only illustrates several general theorems). The holonomy of the connection resulting from parallel transport over a closed curve in state space—yields the famous "geometric phase" of Berry. Note that this connection is invariant under the action of the unitary group on our Hilbert space—it is in fact the only one with this property and will be encountered again in Section 7.

3. Hopf and mechanics—the harmonic oscillator

It may surprise some that the simple two-dimensional isotropic classical harmonic oscillator provides an example of the occurrence of the Hopf fibration in physics. But indeed, its phase flow, restricted to a non-trivial energy shell, Hopf-fibers the latter! To see this easily, we scale everything such that the mass and the angular frequency of the oscillator become 1; the Hamiltonian then reads

$$H = \tfrac{1}{2}(p_1^2 + p_2^2 + q_1^2 + q_2^2),$$

whence the equations of motion

$$\dot{q}_k = p_k, \qquad \dot{p}_k = -q_k \quad (k = 1, 2).$$

Employing the usual complex variables $a_k := q_k + \mathrm{i}\,p_k$, these become

$$H = \tfrac{1}{2}(|a_1|^2 + |a_2|^2), \qquad \dot{a}_k = -\mathrm{i}a_k$$

with solution $a_k(t) = a_k(0)\exp(-\mathrm{i}t)$. If $H(0) \neq 0$, energy conservation allows to normalize $a_k/\sqrt{2H} =: z_k$, the z_k satisfying $z_k(t) = z_k(0)\exp(-\mathrm{i}t)$, $|z_1|^2 + |z_2|^2 = 1$. Hence we see that the phase space trajectories give a Hopf fibration of the energy shell. Moreover, the 2-surfaces given by $H_1 = \text{const.}$, $H_2 = \text{const.}$, where $H_k = \frac{1}{2}(p_k^2 + q_k^2) = \frac{1}{2}|a_k|^2$ are conserved and have vanishing Poisson bracket with each other, are analogous to the tori whose stereographic projection is depicted in [48]; they are just the Arnold tori of our integrable system. The similarity of the pictures of the Hopf fibration with Fig. 242 of [1] or with Fig. 31 of [16] is, therefore, not accidental. More details, such as the relation of the Hopf map to the concept of moment map, as well as its role in rigid body motion and classical analogs of the quantum geometric phase, together with many historical remarks, are given in Section 1.10 of [51].

Interesting generalizations concerning the linking of flux lines of divergence-free vector fields on 3-manifolds, with application, e.g., as an obstacle to the dissipation of magnetic energy in stars, discussed in Chapter III of [2], show that the important feature of the Hopf fibration in the present context is the linking of its fibers (cf. Appendix A).

4. Hopf and General Relativity—Taub-NUT space

Around 1950, Gödel and Taub independently started the investigation of spatially homogeneous cosmological models in General Relativity. The ones constructed by them are space-times **M** having line elements of the form

$$\mathrm{d}s^2 = -\mathrm{d}\tau^2 + \gamma_{ik}(\tau)\omega^i\omega^k,$$

where the ω^k are essentially left-invariant 1-forms on a three-dimensional Lie group $\mathcal{G}$ and the product of forms in $\mathrm{d}s^2$ is symmetric tensor multiplication as usual. More precisely, $\mathbf{M} = \mathbf{I} \times \mathcal{G}$, where $\mathbf{I}$ is some interval for τ, to be determined during the process of solving the Einstein field equations, which become ODEs under the ansatz above, and the ω^k are pullbacks under $\mathbf{M} \to \mathcal{G}$ of left-invariant forms on $\mathcal{G}$.

By "the" Taub universe one means the solution of Einstein's vacuum field equations with vanishing cosmological constant in which the ω^k belong to $\mathcal{G} = \mathrm{SU}(2)$ ("Bianchi type IX") and may be constructed from the manifestly left-invariant Maurer–Cartan matrix $U^{-1}\,\mathrm{d}U$, which is Lie algebra-valued and so may be decomposed as $(-\mathrm{i}/2)\omega^k\sigma_k$, the σ_k again being the Pauli matrices. Further, $\gamma_{ik}(\tau)$ is assumed diagonal and $\gamma_{11} = \gamma_{22}$. As a consequence, $\mathrm{d}s^2$ has as isometries not only the left translations of SU(2) (augmented by $\tau \to \tau$) but also the 1-parameter group of right translations generated by $-\mathrm{i}\sigma_3$: the latter does not preserve ω^1, ω^2 but does preserve ω^3 and $(\omega^1)^2 + (\omega^2)^2$. Using the group approach to the Hopf fibration (described in Appendix A), we see that it is this extra symmetry which brings in the Hopf fibration—and it also permits the Einstein equations to be solved explicitly. (Without it, the SU(2) model is known as "mixmaster universe" [30], for which the vacuum field equations are not integrable; on the other hand, requiring $\gamma_{11} = \gamma_{33}$ in addition implies symmetry under all right translations and thus makes the model isotropic as well, so that no distinguished Hopf fibration appears: this is the geometry of the standard Robertson–Walker cosmology with positive spatial curvature.) Even without imposing field equations, the extra symmetry causes the 4-geometry to be "algebraically special" in that its Weyl tensor is of Petrov–Penrose type D [11,35], both degenerate principal null directions of it having the Hopf fiber tangents for their spatial projections. This is also related to the "optical geometry" of the Taub space-time, which in turn is related to the standard "CR-structure" of $\mathbf{S_3}$ [46]. When the vacuum field equations (without cosmological constant) are imposed, the general solution of the resulting system of ODEs is [40]

$$\gamma_{11} = \frac{k}{2}\frac{\cosh\,(kt_{\mathrm{Taub}} + \alpha)}{1 + \cosh\,(kt_{\mathrm{Taub}} + \beta)} = \gamma_{22}, \qquad \gamma_{33} = \frac{k}{\cosh\,(kt_{\mathrm{Taub}} + \alpha)},$$

$$\mathrm{d}\tau^2 = \gamma_{11}^2\gamma_{33}\,\mathrm{d}t_{\mathrm{Taub}}^2.$$

Here α, β and $k > 0$ are the constants of integration.[2] The interval for t_{Taub} is the whole real line, but contrary to Taub's original belief this does not prevent this space-time from being incomplete. We cannot go into a discussion of this in any detail, but only highlight the appearance of the Hopf fibration in the further development. By a number of transformations one (analytically yet non-uniquely!) continues to other regimes (NUT regimes, after [28,34])—remarkably discovered independently, following an entirely different route), where the symmetries described above—which are spacelike in the Taub regime—become timelike in part. More specifically, it is the Killing field tangent to the Hopf fibers which becomes timelike; thus the appearance of closed timelike lines makes the solution very peculiar. The transitions from one regime to the other happens [32,35]) across a "Cauchy horizon", an $\mathbf{S_3}$ null hypersurface generated by the SU(2) symmetry group, whose null geodesic generators are just Hopf fibers.[3] In the NUT regime, a further feature of the Hopf fibration becomes important—the linking of the fibers. Generally, the linking of the inverse images of points under a smooth map $\mathbf{S_3} \to \mathbf{S_2}$ can be expressed by a certain integral, its Hopf invariant [4], and it turns out that this integral is related here to a physical quantity called "NUT charge" or "dual mass" [38]. The linking also prevents the NUT regime from being asymptotically flat in the usual global sense, although its curvature tensor goes to zero asymptotically: it is only locally asymptotically flat, there are no global spacelike hypersurfaces. Nevertheless a null conformal boundary $\mathcal{I}$ may be attached [38] to the asymptotic region; however, it does not have the usual topology $\mathbf{S_2} \times \mathbf{R}$ but rather $\mathbf{S_3}$, again Hopf-fibered by its null generators. There are many more peculiar properties of the Taub-NUT solution which we cannot even mention here [29]. NUT space has been termed the spherically symmetric "gravimagnetic monopole" in [25]; indeed, the analogy to the spherically symmetric Dirac magnetic monopole (Section 7) would become more pronounced if a Kaluza–Klein type formulation (cf. [41]) of the latter were used.

5. Hopf and twistors—Robinson congruences

The concept of twistor can, according to its inventor, Penrose, be introduced in a number of ways, and all conformally invariant field laws in flat space-time can be reformulated in terms of twistors, generating new ways of looking at such laws; this has already been very fruitful in the past in many cases.

Clearly, this is not the place to give any of the different ways of approaching or using twistors. We rather pick out one way of representing a twistor space-time-geometrically

[2] Because it is seldom written down explicitly, we give the transformation between Taub's time coordinate and constants of integration and the ones used later, following the NUT paper [34]. Namely, one achieves a rational dependence on the time coordinate and the constants by introducing t, m, ℓ via

$$\frac{t-m}{\sqrt{m^2+\ell^2}} = \tanh \frac{1}{2}(kt_{\text{Taub}}+\beta), \qquad \frac{m}{\sqrt{m^2+\ell^2}} = \tanh(\alpha-\beta), \qquad 4\ell\sqrt{m^2+\ell^2} = k,$$

rendering $\gamma_{11} = t^2+\ell^2$, $\gamma_{33} = 4\ell^2 U$, $d\tau^2 = dt^2/U$, where $U := (\ell^2+2mt-t^2)/(\ell^2+t^2)$.

[3] Moncrief [31] has shown that there exists a wider class of solutions to Einstein's equations possessing an $\mathbf{S_3}$ Cauchy horizon Hopf-fibered by its null geodesic generators.

and highlight its relation to Hopf, referring to [13,18] for more concerning twistors. For this, we will make a shortcut [47] that, while unfortunately destroying manifest conformal covariance of the geometry, fits nicely to the group geometric description of the Hopf fibration (given in Appendix A). Namely, instead of the group SU(2) considered in Appendix A, we here take the group U(2) of *all* unitary 2×2 matrices. The unitarity restriction for $U \in \mathrm{U}(2)$ implies that the determinant $\det U$ has absolute value 1, so leaves open an arbitrary phase factor for it: U(2) is a four-dimensional compact Lie group. It may in fact be decomposed as a product $\mathrm{U}(1) \times \mathrm{SU}(2)$, but this is slightly tricky, in that the 'naive' decomposition $U = U_1\sqrt{\det U}$ yields a direct product decomposition with $\sqrt{\det U} \in \mathrm{U}(1)$ and $U_1 \in \mathrm{SU}(2)$ only *locally*,[4] since $\sqrt{\det U}$ is 2-valued and it is not possible to pick a unique branch in a continuous fashion because U(2) is not simply connected. No such problem arises if we set

$$U = U_0 \begin{pmatrix} 1 & 0 \\ 0 & \det U \end{pmatrix} \quad \text{or} \quad U = \begin{pmatrix} 1 & 0 \\ 0 & \det U \end{pmatrix} U_0 :$$

the assignments $U \mapsto U_0 \in \mathrm{SU}(2)$ and $U \mapsto \det U \in \mathrm{U}(1)$ are both continuous here. However, the decomposition is now only a *semidirect* one in the algebraic sense: SU(2), being the kernel of the homomorphism $\mathrm{U}(2) \to \mathrm{U}(1)$ given by $U \mapsto \det U$, is an invariant (or normal) subgroup, but the matrices $\begin{pmatrix} 1 & 0 \\ 0 & \det U \end{pmatrix}$ form a subgroup isomorphic to U(1) that is not invariant. So we have the homomorphisms $\det : \mathrm{U}(2) \to \mathrm{U}(1)$ and $\iota_0 : \mathrm{U}(1) \to \mathrm{U}(2)$, sending $\exp \mathrm{i}\alpha \mapsto \begin{pmatrix} 1 & 0 \\ 0 & \exp \mathrm{i}\alpha \end{pmatrix}$, whose composition is the identity on U(1), thus splitting the short exact sequence $1 \to \mathrm{SU}(2) \to \mathrm{U}(2) \to \mathrm{U}(1) \to 1$; this is the situation of a semidirect product. Using this decomposition we see that *topologically*[5] $\mathrm{U}(2) \cong \mathbf{S_1} \times \mathbf{S_3}$.

The relation of this setup to twistor theory comes from a way of putting a Lorentzian global metric on U(2) which is locally conformal to the Minkowski metric and which is such that the cosets of the subgroup $\iota_0(\mathrm{U}(1))$ are *null geodesics*. This system of null geodesics constitutes an example (in fact the only one up to conformal transformations) of what Penrose calls a *Robinson congruence* of null geodesics and which is a geometrical representation—either in U(2) or in the conformally related Minkowski space, as we shall see shortly—of a (projective) *twistor*. We now describe the relation to Hopf and give the metric involved.

[4] Globally, we have a homomorphism of the direct product $\mathrm{U}(1) \times \mathrm{SU}(2)$ onto U(2) given by $(\exp \mathrm{i}\alpha, U_1) \mapsto U_1 \exp \mathrm{i}\alpha$ with kernel $\{(1, \mathbf{1}), (-1, -\mathbf{1}\} \cong \mathbf{Z}_2$. More important for us will be the universal covering homomorphism $\mathbf{R} \times \mathrm{SU}(2) \to \mathrm{U}(2)$, sending $(\alpha, U_1) \mapsto U_1 \exp \mathrm{i}\alpha$ with kernel $\{n\pi, (-\mathbf{1})^n | n \in \mathbf{Z}\} \cong \mathbf{Z}$.

[5] In the group theory sense, one can still regard the 3-sphere as the coset space U(2)/U(1) if U(1) is embedded as just described. This makes it the simplest example of a Stiefel manifold. See [33,42] for physics applications of the more general cases.

The relation to Hopf is easy: one simply applies to the second, global decomposition the first, local one, writing

$$U = U_0 \begin{pmatrix} 1 & 0 \\ 0 & \det U \end{pmatrix} = \underbrace{U_0 \begin{pmatrix} 1/\sqrt{\det U} & 0 \\ 0 & \sqrt{\det U} \end{pmatrix}} \sqrt{\det U},$$

or

$$U = \begin{pmatrix} 1 & 0 \\ 0 & \det U \end{pmatrix} U_0 = \sqrt{\det U} \underbrace{\begin{pmatrix} 1/\sqrt{\det U} & 0 \\ 0 & \sqrt{\det U} \end{pmatrix} U_0}.$$

The unimodular part indicated by the brace constitutes the projection of the coset of $\iota_0(\mathrm{U}(1))$ passing through $U_0 \in \mathrm{SU}(2)$ onto SU(2) in the local direct product decomposition sense. If U_0 is kept fixed while $\det U$ is varied, U traces the coset through U_0 in U(2), while the indicated local projection traces part of the Hopf or anti-Hopf fiber through U_0 in $\mathrm{SU}(2) \cong \mathbf{S_3}$.

To get the metric, start from the right (resp. left) invariant Maurer–Cartan matrix $\Omega := \mathrm{d}U \cdot U^{-1}$ (resp. $= U^{-1}\,\mathrm{d}U$) on U(2); then for arbitrary real constants α, β the symmetric differential forms of degree 2 given by $\alpha(\mathrm{Tr}\,\Omega)^2 + \beta\,\mathrm{Tr}(\Omega^2)$ are bi-invariant, i.e., invariant under $U \mapsto AUB$ for all $A, B \in \mathrm{U}(2)$. (Here multiplication of differentials is symmetric tensor multiplication as in the usual $g_{ik}\,\mathrm{d}u^i\,\mathrm{d}u^k$ for metrics; i.e., if the u^i are some parameters on the group manifold, the g_{ik} for the above forms are $g_{ik} = \alpha\,\mathrm{Tr}(U_{,i}U^{-1})\mathrm{Tr}(U_{,k}U^{-1}) + \beta\,\mathrm{Tr}(U_{,i}U^{-1}U_{,k}U^{-1})$.) Among them, the ones with the ratio $\alpha : \beta = -1$ are of particular importance because of their Lorentzian signature and their high (maximal possible) *conformal* symmetry. If we take $-\alpha = \beta = \frac{1}{2}$, we have

$$\tfrac{1}{2}\mathrm{Tr}(\Omega^2) - \tfrac{1}{2}(\mathrm{Tr}\,\Omega)^2 \equiv -\det\Omega = -\det U^{-1}\det \mathrm{d}U =: \mathrm{d}s_{\mathrm{E}}^2,$$

where also under the det sign symmetric multiplication of differentials is understood.

We refer the reader to [47] or [18] for the description of the full conformal symmetry of $\mathrm{d}s_{\mathrm{E}}^2$ in terms of U, but repeat from there its relation to Minkowski space; it is given as follows. Decompactify U(2) by removing all U having $\det(U - \mathbf{1}) = 0$; to the rest, apply the *Cayley transformation*

$$X = -\mathrm{i}(U + \mathbf{1})(U - \mathbf{1})^{-1} \Leftrightarrow U = (X - \mathrm{i}\mathbf{1})(X + \mathrm{i}\mathbf{1})^{-1},$$

producing Hermitian matrices X from unitary U and conversely. One then calculates

$$\mathrm{d}X = 2\mathrm{i}(U - \mathbf{1})^{-1}\,\mathrm{d}U(U - \mathbf{1})^{-1}, \qquad \det \mathrm{d}X = -4\det U(\det(U - \mathbf{1}))^{-2}\det\Omega.$$

Finally, parametrizing X as

$$X = \begin{pmatrix} x^0 + x^3 & x^1 - \mathrm{i}x^2 \\ x^1 + \mathrm{i}x^2 & x^0 - x^3 \end{pmatrix} = x^0\mathbf{1} + \boldsymbol{x} \cdot \boldsymbol{\sigma}$$

with the Pauli matrices $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$, we see that the Minkowski metric

$$\mathrm{d}s_{\mathrm{M}}^2 := (dx^0)^2 - \mathrm{d}\boldsymbol{x}^2 = \det \mathrm{d}X$$

is conformally related to $\mathrm{d}s_{\mathrm{E}}^2$, i.e., the Cayley transformation is conformal.

The decompactification above removes from the SU(2) subgroup just the single element $\mathbf{1}$; the rest gets, via Cayley, related to the hypersurface $\mathrm{Tr}\, X = 0$—i.e., $x^0 = 0$—of the Minkowski space constituted by the X or (x^i); one checks that this is just a stereographic relation. The system of Hopf fibers on $\mathrm{SU}(2) \cong \mathbf{S_3}$ thus gets mapped to the system of Villarceau circles described in [48].

Let us now have a closer look at $\mathrm{d}s_\mathrm{E}^2$ in order to see that the cosets of our subgroup $\iota_0(\mathrm{U}(1))$ are null geodesics. Write $\det U = \exp 2\mathrm{i}\tau$, $U = U_1 \exp \mathrm{i}\tau$, $U_1 \in \mathrm{SU}(2)$, remembering that τ and $\exp \mathrm{i}\tau$ are only locally defined as smooth functions, having jumps on non-contractible loops which we know exist on U(2); so $\mathrm{d}\tau$ is still globally defined. In terms of this local decomposition, we have $\Omega = \mathrm{i}\,\mathrm{d}\tau\mathbf{1} + \Omega_1$ with $\Omega_1 := \mathrm{d}U_1 \cdot U_1^{-1}$ (resp. $=U_1^{-1}\,\mathrm{d}U_1$), so that $\mathrm{Tr}\,\Omega_1 = 0$ from $\det U_1 = 1$. This gives the manifestly global definition of $\mathrm{d}\tau$ as $(1/2\mathrm{i})\mathrm{Tr}\,\Omega$ and

$$\mathrm{d}s_\mathrm{E}^2 = -\det(\mathrm{i}d\tau\mathbf{1} + \Omega_1) = \mathrm{d}\tau^2 - \det \Omega_1,$$

where $\det \Omega_1 \equiv \det \mathrm{d}U_1$ is seen (Appendix A) to be the usual line element on $\mathbf{S_3} \cong \mathrm{SU}(2)$, for which the Hopf fibers, being great circles, are geodesics. Since $\mathrm{d}s_\mathrm{E}^2$ is a Riemannian sum of two metrics, its geodesics are given by their local projections which are geodesics in their respective metrics. Thus the cosets under consideration are indeed geodesics for $\mathrm{d}s_\mathrm{E}^2$; and they are null, since parametrizing them as $\alpha \mapsto U_0 \begin{pmatrix} 1 & 0 \\ 0 & \exp \mathrm{i}\alpha \end{pmatrix}$ (resp. $\alpha \mapsto \begin{pmatrix} 1 & 0 \\ 0 & \exp \mathrm{i}\alpha \end{pmatrix} U_0$), we easily find $\det \Omega = 0$ along them.

Two remarks are as follows. (1) On passing from U(2) to its universal covering group $\mathbf{R} \times \mathrm{SU}(2)$ (cf. the preceding footnote), τ becomes globally defined, and $\mathrm{d}s_\mathrm{E}^2$ is then the usual metric on the Einstein cylinder universe $\mathbf{R} \times \mathbf{S_3}$ (up to some scaling); and while the closure of the Cayley image of Minkowski space within U(2) is all of U(2), the closure (of a connected lift) within the universal cover gives the usual 'conformal infinity' picture [13]. (2) Performing the local projection onto SU(2) first and Cayley afterwards is not the same as doing Cayley first and orthogonally projecting onto SU(2)'s Cayley image 3-space $x^0 = 0$ in Minkowski space thereafter. Thus the Cayley images of the above cosets (null geodesics of $\mathrm{d}s_\mathrm{E}^2$) are null geodesics of $\mathrm{d}s_\mathrm{M}^2$ and therefore straight null lines by the well-known conformal invariance property of null geodesics. So their Minkowski orthogonal projections onto $x^0 = 0$ are straight lines there and not the stereographic Hopf = Villarceau circles! As Penrose points out, the latter are useful nevertheless in this context, however: a null straight line in Minkowski space is easily constructed from an initial point in $x^0 = 0$ and the direction there of its projection onto $x^0 = 0$ by running through the straight line given by these data with the speed of light; and for the null geodesics in question, *this* direction is given by the tangent to the stereographic Hopf circle at the stereographic image point. This is how Penrose instructs us to read the stereographic Hopf fibration in the twistor context.

We finally describe, following Penrose, the relation to (projective) twistors. Twistors are elements of a four-dimensional complex vector space $\mathbf{T}$ equipped with a Hermitian form h of signature $(2, 2)$ and a compatible determinant function ϵ; the automorphism group of this structure, SU(2, 2), is a 4:1 covering of the (connected component of the) conformal group of ds^2_E. We will be interested in the projective twistors, i.e., the one-dimensional subspaces $\mathbf{C}T$ of $\mathbf{T}$ ($0 \neq T \in \mathbf{T}$)—so notice that all equations below are homogeneous in the twistor variables that occur, unless special normalizations are made. We first interpret $\mathbf{C}N$, $0 \neq N \in \mathbf{T}$, in the null case $h(N, N) = 0$, and only then interpret $\mathbf{C}T$, $0 \neq T \in \mathbf{T}$, in the non-null case $h(T, T) \neq 0$ by looking at those null twistors $N \neq 0$ which are orthogonal to T, $h(N, T) = 0$: this will lead to the Robinson congruences.

We choose a basis in $\mathbf{T}$ that diagonalizes h, so that its matrix is $\mathrm{diag}(1, 1, -1, -1)$; then $h(T, N) = \overline{T^1}N^1 + \overline{T^2}N^2 - \overline{T^3}N^3 - \overline{T^4}N^4$. Consider now a null twistor N: we have $|N^1|^2 + |N^2|^2 = |N^3|^2 + |N^4|^2$. Therefore a unitary 2×2 matrix $U \in \mathrm{U}(2)$ exists[6] effecting $U \begin{pmatrix} N^3 \\ N^4 \end{pmatrix} = \begin{pmatrix} N^1 \\ N^2 \end{pmatrix}$, but it is not unique. We exemplify this by the choice $\begin{pmatrix} N^3 \\ N^4 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$: we then have $|N^1|^2 + |N^2|^2 = 1$, and U is given by $U = U_0\,\mathrm{diag}(1, \exp \mathrm{i}\alpha)$ with $U_0 = \begin{pmatrix} N^1 & -\overline{N^2} \\ N^2 & \overline{N^1} \end{pmatrix} \in \mathrm{SU}(2)$ and α real—so in this case $\mathbf{C}N$ determines and is determined by the left coset through U_0 of $\iota_0(\mathrm{U}(1))$. For some other choice of N^4, we would similarly have obtained the coset of a conjugate subgroup, which, by the invariance of our metric, would again constitute a null geodesic. This gives the interpretation of null twistors. The choice $N^4 = 0$ of the example already exemplifies the interpretation of a non-null twistor T having $h(T, T) < 0$, since if we take T to have T^4 as its only non-zero component, the condition $h(T, N) = 0$ just implies $N^4 = 0$, allowing to normalize N to look as above with $U_0 \in \mathrm{SU}(2)$ arbitrary. Thus $\mathbf{C}T$ indeed gets interpreted by all the cosets of $\iota_0(\mathrm{U}(1))$, constituting the Robinson congruence considered before. Similarly, if we choose T to have T^2 as its only non-zero component, exemplifying non-null twistors having the opposite sign $h(T, T) > 0$, the null twistors N orthogonal to it satisfy $N^2 = 0$; when we scale $N^1 = 1$, the condition on U becomes the same as before with U replaced by U^{-1} and N^1, N^2 by N^3, N^4, implying that U now runs through right rather than left cosets: this is related to the anti-Hopf fibration, having the opposite sense of twist. (A general position of T leads to a general position of the congruence, whose description in Minkowski space by the method above would involve Dupin cyclides instead of tori.)

6. Hopf and Wignerism—helicity representations

In all examples discussed so far, the bundle, or total, space $\mathbf{S_3}$ was of immediate importance and thus in the forefront of the formalism; the base space $\mathbf{S_2}$, on the other hand, did figure in the qubit case and is used in technical treatments of NUT space that work with the

[6] The description of conformal transformations in terms of fractional linear block matrix transformations of the unitary matrices U as given in [47] can be directly inferred from this parametrization of null twistors. A similar parametrization using Hermitian matrices X is possible when components with respect to a *null* basis are used (the relation of the X to the U being given just by the Cayley transformation); cf. [18,13].

space of Killing trajectories. In the two examples to follow, it will be the other way round: it is the base space that is in the foreground, while the bundle space appears as derived, e.g., via a patching construction. The latter is necessitated by the need to look at the global aspects of the objects involved. The situation in which such an object is a connection or covariant derivative will be treated in the next section; common to that example and the one to be treated now is a smoothness requirement that will provide a domain of definition for certain differential operators.

In the present example, the latter will be the generators of the (homogeneous, proper, orthochronous) Lorentz group $\mathcal{L}$ in the well-known "massless" *helicity representations* of the (universal covering group $\tilde{\mathcal{P}}$ of the proper orthochronous) Poincaré group $\mathcal{P}$. It may surprise some that something topologically unusual should be lurking behind this matter, but consequences of this fact have indeed been discussed in the literature [21,22]. These representations show up as one of the "physical" cases in Wigner's classification of the unitary irreducible representations of $\tilde{\mathcal{P}}$, which in turn are building blocks of any formalism that realizes relativistic symmetry in the quantum domain [49].

From the fact that the translation subgroup $\mathcal{T}$ of $\mathcal{P}$ is an Abelian invariant subgroup one derives that every such representation can be realized in the space $\Gamma(E)$ of sections of a (complex) vector bundle E over an orbit $\mathbf{O}$ of $\mathcal{L}$ in the vector space $\mathcal{T}^*$ dual to $\mathcal{T}$. (In physics, that vector space is the space of wave numbers and is, according to de Broglie, via $\times$ (inverse Minkowski metric tensor), identified with $\mathcal{T}$ except for its physical dimension, which then is momentum.) $\tilde{\mathcal{L}}$ (covering $\mathcal{L}$) acts on E, each $\tilde{L} \in \tilde{\mathcal{L}}$ transforming the fiber over $p \in \mathbf{O}$ into the fiber over Lp by a linear transformation $Q(\tilde{L}, p)$. From this derives a linear action on $\Gamma(E)$ which is irreducible iff for any "origin" $\bar{p} \in \mathbf{O}$ the $Q(\tilde{L}, \bar{p})$ furnish a unitary irreducible representation when the $\tilde{L}$ are restricted to the subgroup $\tilde{\mathcal{K}}_{\bar{p}}$—the "little group"—of $\tilde{\mathcal{L}}$ that leaves $\bar{p}$ unchanged. Further, the representation of $\tilde{\mathcal{L}}$ in $\Gamma(E)$ is determined ("induced") uniquely up to unitary equivalence by that representation of the little group. Thus the classification is by classifying the orbits of $\mathcal{L}$ in 4-vector space and by classifying the unitary irreducible representations of the associated little groups.

The case of interest for us is the one where $\mathbf{O}$ is a half null cone in $\mathcal{T}^*$ (forward or backward), corresponding to zero mass particles, so that $\bar{p}$ is null (lightlike) and the little group has structure $\mathrm{U}(1) \times \mathbf{C}$ (semidirect product). Further, the relevant representations of the little group are those where the invariant subgroup corresponding to the factor $\mathbf{C}$ (covering "null rotations") is represented trivially in the fiber over $\bar{p}$; then, for irreducibility and unitarity, the U(1) factor has to be represented one-dimensionally, i.e., the whole representation is determined by a character of U(1), and the vector bundle has one-dimensional fibers—it is a complex line bundle. The characters are indexed by integers; since U(1) double-covers a 1-parameter subgroup of rotations of $\mathcal{L}$ and since the effect of non-null translations on the state vectors of the representation may be compensated by such rotations depending on this integer, 1/2 times the relevant integer is called the *helicity* λ of the particle associated with the representation.

Our interest in those representations comes from the fact that for non-vanishing λ the line bundle is topologically non-trivial, enabled by the fact that the base space null cone $\mathbf{O} \cong \mathbf{S}_2 \times \mathbf{R}$ is non-contractible. The interesting topology already resides in the restriction of the line bundle to an $\mathbf{S_2}$ cross-section of the null cone $\mathbf{O}$; we may take it to be the orbit of

$\bar{p}$ under the SO(3) subgroup of $\mathcal{L}$ and may then restrict the little group to be U(1). We thus get an induced representation[7] of SU(2) (double-covering SO(3)), carried by the space of sections of the restricted line bundle over $\mathbf{S_2}$, and this is the point where the Hopf fibration appears in the setup.

To see this first as cheaply as possible, we use a mathematical trick of the trade ([9, p. 114]; [41]). Namely, in situations like the one above ("homogeneous vector bundles"), there is a standard bijection between sections ψ of vector bundles over group orbits and vector-valued functions $\tilde{\psi}$ on the group itself with a prescribed behavior along the cosets of the little group.[8] In our case as restricted to $\mathbf{S_2}$ and SU(2), to every section of the line bundle there corresponds bijectively a $\mathbf{C}$-valued function $\tilde{\psi}$ on SU(2) whose values change as $\tilde{\psi}(U \exp \mathrm{i}\alpha\sigma_3) = \exp(2\mathrm{i}\lambda\alpha)\tilde{\psi}(U)$ as one goes along a left coset of U(1) in SU(2). Thus again using the group description of the Hopf fibering, we see that the Hopf fibers serve as "guiding lines" for the behavior of the functions $\tilde{\psi}$. The representation of the $\tilde{L}$ is, in this language, given by composing the $\tilde{\psi}$ with the left translation by $\tilde{L}$. (Since the Hopf fibration is left-invariant, the defining property of the $\tilde{\psi}$ is not lost.)

In physics texts, this is presented somewhat differently and slightly heuristically. One uses the system of "improper" eigenvectors common to all translation operators as a basis in representation space. They are indexed in part by the spectral values of the translation generators, and the Lorentz group acts on that spectrum which thus must, for irreducibility, be an orbit of the Lorentz group. One fixes $\bar{p}$ and chooses a basis in the eigenspace indexed by it. One then imagines this basis transported to all other eigenspaces, using the representing operators of selected Lorentz transformations, one for each orbit point p to be reached from $\bar{p}$. Traditionally, these coset representatives are chosen, for definiteness and explicitness, as boosts. In the "massive" case, $\bar{p}$ is timelike, and the boost is taken with respect to an "observer" whose 4-velocity is parallel to it; but in the massless case we are interested in, such choice is impossible, since a 4-velocity must be timelike. However, if any timelike choice of the observer is made, there is a unique boost with respect to it that carries $\bar{p}$ into p (by the Doppler and aberration effect, physically speaking) *except* when p has its space direction antipodal to that of $\bar{p}$ with respect to the observer. In this case, the boost becomes undefined, the necessary speed being that of light. One might argue that fixation to boosts w.r.t. one single observer or to any boost at all is unnecessary, it could be just any Lorentz transformation representing the coset defined by p. But for a simple topological reason no choice that would depend continuously on p is possible for all of $\mathbf{O}$, because if it were, the topology of $\tilde{\mathcal{K}}_{\bar{p}}$ and of $\mathbf{O}$ would imply the infinitely connected topology $\mathbf{S_2} \times \mathbf{R} \times \mathrm{U}(1) \times \mathbf{R^2}$ for $\tilde{\mathcal{L}}$—but the latter is simply connected. Why bother about continuity at a single null line

[7] As a representation of SU(2), it is reducible; according to Frobenius reciprocity, it contains all irreducible representations with highest weights $\lambda, \lambda+1, \lambda+2, \ldots$ exactly once, and none else. The irreducible subspaces are spanned, in the "ψ-language" below—by "spin-weighted spherical harmonics" [12], in the context of Section 7 also known as "monopole spherical harmonics". Incidentally, since the base space $\mathbf{S_2}$ may be regarded as the complex manifold $\mathbf{CP_1}$, it makes sense to ask whether these complex line bundles are holomorphic—and they are. The spaces of their *holomorphic* sections are only finite-dimensional, as one may check by series expansion around the north and south pole, and are easily related to binary forms in the homogeneous coordinates z_1, z_2 of $\mathbf{CP_1}$ which are well known to carry the irreducible representations of SU(2). This illustrates a famous general construction of Borel–Bott–Weil in rudimentary form.

[8] Explicitly, if $\psi : \mathbf{O} \to E$ is the section, the associated $\tilde{\psi}$ is given by $\tilde{\psi}(\tilde{L}) = (Q(\tilde{L}, \bar{p}))^{-1}\psi(L\bar{p})$.

on the cone $\mathbf{O}$—in the end the representation space $\Gamma(E)$ will be enlarged anyway to include all sections square-integrable over the cone? The reason is that one wants to have continuous representations, and for them one must provide a domain of definition for the infinitesimal generators of $\tilde{\mathcal{L}}$, which are differential operators—so we want to start with the space of smooth sections of a smooth vector bundle. The above continuous, in fact smooth, choice allows the construction of the bundle just over the patch of $\mathbf{O}$ where the boosts work, and one can then choose an anti-podal origin and boosts "starting" from there to obtain a second patch, and finally match the two. The bundle so obtained is non-trivial for non-zero helicity, but it is clear now that the trick used above is preferable to doing the patching explicitly: this is possible in our "group-dominated" situation. We just mention that for the restricted bundle over $\mathbf{S_2}$ considered above one can choose the coset representatives to belong to SO(3) instead of being boosts; one then encounters a similar problem, and the necessary patching will be related to the patching to be treated in our next example.

Alternatively, one can see the occurrence of the Hopf fibration by starting from the Weyl equation $(\partial_t \pm \boldsymbol{\sigma} \cdot \nabla)\psi = 0$ and the "positive frequency condition" for 2-component spinor fields, which are well known to carry helicity $\pm\frac{1}{2}$ under these conditions. In Fourier (i.e., momentum) space, the latter are $(p^0\mathbf{1} \pm \boldsymbol{p} \cdot \boldsymbol{\sigma})\tilde{\psi}(p) = 0$ and $p^0 > 0$, implying that the Hermitian matrix $2P := p^0\mathbf{1} \pm \boldsymbol{p} \cdot \boldsymbol{\sigma}$ is singular. This puts p on the forward light cone $\mathbf{O}$ and implies P to be of rank 1, so $P = zz^\dagger$ for some complex 2-rowed column z (2-component spinor). It follows then that $z^\dagger\tilde{\psi} = 0$, so $\tilde{\psi}$ is uniquely determined by z apart from a complex factor, while z is uniquely determined by p apart from a phase factor. This says that $p \mapsto \tilde{\psi}(p)$ is a section of a complex line bundle over $\mathbf{O}$, associated to $\mathbf{C^2} \setminus \{0\} \cong \mathbf{S_3} \times \mathbf{R}$ viewed as a circle bundle over $\mathbf{O} \cong \mathbf{S_2} \times \mathbf{R}$ whose fibers over the p consist of all $z \in \mathbf{C^2}$ having $zz^\dagger = P$. So if we consider the + case and restrict to the cross-section $p^0 = 1$ of the cone, we can immediately compare P formally with the 2×2 density matrix of a pure state as discussed in Section 2 to obtain the same geometry as discussed there.

The − sign leads to anti-Hopf; other helicities obtain from generalized Weyl equations for positive frequency fields carrying $2|\lambda|$ totally symmetric indices [12]; the corresponding line bundles over $\mathbf{O}$ are then tensorial powers of the one above—the tautological line bundle of $\mathbf{CP_1}$, cf. Appendix A—and of its dual, as follows easily from the spinor algebra described in [12]. This will bring in identifications modulo $2|\lambda|$th roots of unity, related to so-called "lens spaces" (Appendix A), also to be mentioned in the next section.

7. Hopf and magnetic monopoles

This example of our collection is probably the first one that caught the attention of a larger number of physicists. As announced, the Hopf fibration comes here from a bottom-up construction.

In his 1931 article "Quantised Singularities in the Electromagnetic Field" (magnetic monopoles), Dirac [19] starts out looking for a generalization of wave mechanics in which the wave function does not have a definite value for its phase difference between two points. In the 1970s, it was realized [50] that it is advantageous to rephrase Dirac's idea in the "fiber bundle with connection" language that had been given to electromagnetism and non-Abelian gauge theories in the 1960s by DeWitt [20], Lubkin [27], Loos [26], Trautman [41]. In the e.m. case and neglecting spin and inner degrees of freedom, the relevant bundle is a complex line bundle E over space-time, and instead of ordinary $\mathbf{C}$-valued wave functions one considers sections of the line bundle. (Dirac still called them wave functions.) Ordinary numerical wave functions reappear when a basis (consisting of only *one* non-zero vector per fiber in the case of a line bundle!) is chosen in each fiber, smoothly from point to point, and the value of the section at each point is written as a (complex) scalar multiple of that basis vector. The gain in generality here is that one can consider non-trivial line bundles ("twisted" like Moebius bands), which are still smooth but not capable of a smooth choice of basis vectors, except locally. In regions where two choices (two "gauges") coexist, one has two smooth numerical wave functions for one smooth section, related by a smooth gauge transformation (of "second kind"); but although a section can be smooth globally, in the non-trivial case there is no globally smooth choice of gauge (due to the independence condition on basis vectors), and thus no globally smooth numerical wave function for that section. To include the cases where the line bundle has, because of spin and inner degrees of freedom, to be replaced by a vector bundle with higher fiber dimension, one isolates from the line bundle the bundle of its unit vectors—a circle bundle over space-time, a principal fiber bundle with structure group U(1)—and then considers vector bundles "associated" to it. In this scheme, there is a global but more abstract description of the e.m. field in terms of a connection and its curvature on the (circle) bundle space, and a local "patchwork" description of it in terms of the usual scalar and vector potentials.

To see in more detail how the Hopf fibration relates to this, we first indicate how to derive the Dirac quantization condition on magnetic charges in the modern approach. To the magnetic field $\boldsymbol{B}$ produced by a static distribution of (hypothetical) magnetic charge, compactly supported, with total magnetic charge $g \neq 0$, there corresponds the equivalent 2-form $\beta = \boldsymbol{B}\,\mathrm{d}\boldsymbol{S} = B_x\,\mathrm{d}y \wedge \mathrm{d}z + \cdots$ having $\operatorname{div}\boldsymbol{B} = 0$ or $d \wedge \beta = 0$ outside the support. Locally, we then have there the existence of a vector potential $\boldsymbol{A}$ or the equivalent 1-form $\alpha = \boldsymbol{A}\,\mathrm{d}\boldsymbol{x}$ such that $\boldsymbol{B} = \operatorname{curl}\boldsymbol{A}$ or $\beta = d \wedge \alpha$. If one restricts β, α to a sphere $\mathbf{S_2}$ surrounding the support and uses the same symbols for the restrictions, one still has locally $\beta = d \wedge \alpha$; but α cannot exist globally on the sphere, as this would give $0 \neq g = \int_{\mathbf{S_2}} \boldsymbol{B}\,\mathrm{d}\boldsymbol{S} = \int_{\mathbf{S_2}} \beta = \int_{\partial \mathbf{S_2}} \alpha = 0$ by Stokes, since the sphere is closed. Covering the sphere by $\mathbf{U}^+ := \mathbf{S_2} \setminus \{\text{south pole}\}$, $\mathbf{U}^- := \mathbf{S_2} \setminus \{\text{north pole}\}$, one has for the restrictions $\beta^\pm$ of β to $\mathbf{U}^\pm$ that $\beta^\pm = d \wedge \alpha^\pm$ with 1-forms $\alpha^\pm$ defined *all* over $\mathbf{U}^\pm$, since these are homeomorphic to $\mathbf{R}^2$. Clearly on $\mathbf{U} := \mathbf{U}^+ \cap \mathbf{U}^-$ we have $0 = \beta^+ - \beta^- = d \wedge (\alpha^+ - \alpha^-)$, so locally there $\alpha^+ - \alpha^- = \mathrm{d}\Lambda$ for some function Λ. Since $\mathbf{U}$ is not simply connected, Λ will not exist globally there as a continuous function. For example, if we consider Λ along the equator contained in $\mathbf{U}$, it will have a jump somewhere on it of size $\oint \mathrm{d}\Lambda = \oint \alpha^+ - \oint \alpha^- = \int_{\mathbf{H}^+} d \wedge \alpha^+ + \int_{\mathbf{H}^-} d \wedge \alpha = \int_{\mathbf{H}^+} \beta^+ + \int_{\mathbf{H}^-} \beta = \int_{\mathbf{S_2}} \beta = g$, using Stokes and denoting by $\mathbf{H}^\pm$ the hemispheres contained in $\mathbf{U}^\pm$. (Note that this involves a convention on the unit for g.) The replacement, on $\mathbf{U}$, of α^- with $\alpha^+ = \alpha^- + \mathrm{d}\Lambda$ is called a local *gauge transformation*.

Now quantum mechanics comes in, where the Hamiltonian $H = (1/2m)\boldsymbol{p}^2$ entering the Schrödinger equation for a particle with electrical charge e moving in a given magnetic field is built from the canonical momentum $\boldsymbol{p} = (/\mathrm{i})\nabla + (e/c)\boldsymbol{A}$, which is gauge-dependent. Therefore, a gauge transformation of the local potential $\boldsymbol{A}^-$ has to be accompanied by a phase transformation ("gauge transformation of second kind") of the numerical wave function ψ^- of the particle, i.e., by its replacement with $\psi^+ = \psi^- \exp(-\mathrm{i}(e/c)\Lambda)$, all on $\mathbf{U}$, to keep the physics locally gauge-invariant. However, $\psi^\pm$ have to be smooth there, being restrictions of functions belonging to a domain of definition of the Hamiltonian, which is a differential operator. So, although globally Λ has the non-zero jump g somewhere on the equator, its exponential $\exp(\mathrm{i}(e/c)\Lambda)$ is not permitted to have one, which implies $(eg/2\pi c) \in \mathbf{Z}$ to be an integer n. This is the Dirac condition. The integer n may be interpreted as the winding number, or mapping degree, of the map from the equator circle to the unit circle given by $\boldsymbol{x} \mapsto \exp(\mathrm{i}(e/c)\Lambda(\boldsymbol{x}))$.

With the idea of interpreting $\psi^\pm$ as component functions of a smooth section ψ of a smooth complex line bundle with respect to two local normalized basis sections $s^\pm$, one derives the transition function between the latter: $s^+ = \exp(\mathrm{i}(e/c)\Lambda)s^-$. From the interpretation of $(d + (\mathrm{i}e/c)\alpha^\pm)\psi^\pm$ as the local components of a covariant differential of a section ψ one gets the interpretation of $(\mathrm{i}e/c)\alpha^\pm$ as local pullbacks $\Gamma_\pm$ under $s^\pm$ of a connection form ω on a principal bundle whose curvature form is $(\mathrm{i}e/c)\boldsymbol{B}\,\mathrm{d}\boldsymbol{S}$. Multiplying by $-(2\pi\mathrm{i})^{-1}$, one gets [10, p. 309] the representative $-(e/2\pi c)\boldsymbol{B}\,\mathrm{d}\boldsymbol{S}$ of its Chern class and $-(eg/2\pi c) = -n$ as its *Chern number*. The integer ("topological quantum number") n, being the winding number of the transition map taken along the equator, characterizes a complex line bundle over $\mathbf{S_2}$ or the associated circle bundle of its unit vectors uniquely up to bundle isomorphisms [4, p. 301]. It should be clear now that from the conceptual point of view the state vectors of our quantum mechanical system are given by global sections of the relevant line bundle and that the Hamiltonian is built from the covariant derivative operators acting on sections.

Our point here is that for $n = 1$ such a circle bundle is isomorphic to the Hopf bundle. ($n = -1$ would correspond to anti-Hopf.) Since we have defined the Hopf bundle up to now globally as embedded into $\mathbf{R^4}$, to see this we must first patch the bundle up as above and find two sections, one smooth over $\mathbf{U}^+$, the other smooth over $\mathbf{U}^-$, and then look at the transition function between them when followed round the equator of the basis space $\mathbf{S_2}$. A section s is the result of trying to find a right inverse to the bundle projection $z \mapsto \boldsymbol{R}(z)$. Given $\boldsymbol{R} \in \mathbf{S_2} \subset \mathbf{R^3}$, we form $\rho = \frac{1}{2}(\mathbf{1} + \boldsymbol{R}\cdot\boldsymbol{\sigma})$, which is Hermitian-positive with trace 1 and rank 1; so $\rho = zz^\dagger$ for some $z \in \mathbf{C^2}$ having $z^\dagger z = 1$, unique up to a phase factor—but we know we cannot make the choice such that z is a continuous function of $\boldsymbol{R}$ all over $\mathbf{S_2}$. Using polar coordinates for $\boldsymbol{R}$, we may take $z = s^+(\theta,\phi) = \begin{pmatrix} \cos(\theta/2) \\ \sin(\theta/2)\mathrm{e}^{\mathrm{i}\phi} \end{pmatrix}$ for a section s^+ continuous over $\mathbf{U}^+$ and $z = s^{-1}(\theta,\phi) = \begin{pmatrix} \cos(\theta/2)\mathrm{e}^{-\mathrm{i}\phi} \\ \sin(\theta/2) \end{pmatrix}$ for a section s^- continuous over $\mathbf{U}^-$. (Since $\theta = 0$ and $0 = \pi$ each give only one point on the sphere for all ϕ, these sections are discontinuous at the south and north pole, respectively; see Appendix A for the less dangerous complex stereographic parametrization.) The transition function thus equals $\mathrm{e}^{\mathrm{i}\phi}$, and the associated winding number as defined above is 1, which we wanted to check.

Actually, the Hopf fibration provides more than just the topology ([42]; [17, p. 110]). Namely, the canonical connection form on the bundle space, given globally by $z^\dagger\,\mathrm{d}z$ and locally, using a parametrization given in Appendix A, by $(\mathrm{i}/2)(\mathrm{d}\psi - \cos\theta\,\mathrm{d}\phi)$ is invariant under the U(2) action on z and pulls back by the sections $s^{\pm}$ (which are obviously given by $\psi = \pm\phi$) to the forms $(\mathrm{i}/2)(\pm 1 - \cos\theta)\,\mathrm{d}\phi$ with curvature form (exterior derivative) $(\mathrm{i}/2)\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi \equiv (\mathrm{i}/2)(\boldsymbol{x}/r^3)\,\mathrm{d}\boldsymbol{S}$. These provide the vector potentials $\boldsymbol{A}^{\pm}$ and magnetic field $\boldsymbol{B}$ for the exterior of a magnetic charge distribution invariant under the associated SO(3) action and thus spherically symmetric—e.g., of a single point monopole; one immediately checks that the Chern number is -1. To construct the circle bundle globally outside the point monopole, i.e., over the part $\mathbf{R}^3 \setminus \{0\} \cong \mathbf{S}_2 \times \mathbf{R}$ of *physical* space $\mathbf{R}^3$, we present the bundle space as $\mathbf{C}^2 \setminus \{0\} \cong \mathbf{R}^4 \setminus \{0\} \cong \mathbf{S}_3 \times \mathbf{R}$ (rather than only its unit sphere) but factor out only the U(1) action $z \mapsto \mathrm{e}^{\mathrm{i}\alpha} z$ (rather than the $\mathbf{C}^{\times}$ action, which again would give only the quotient $\mathbf{S}_2 \cong \mathbf{CP}_1$). Concretely, the bundle projection is $z \mapsto z^{\dagger}\boldsymbol{\sigma} z$; all concentric spheres get Hopf-fibered this way as in Section 3. A U(2)-invariant connection form on this U(1) bundle is given by $\mathrm{i}\Im(z^{\dagger}\,\mathrm{d}z/z^{\dagger}z) \equiv (i/2)(\mathrm{d}\psi - \cos\theta\,\mathrm{d}\phi)$, where now θ, ϕ are regarded as angular coordinates on all of $\mathbf{R}^3 \setminus \{0\}$. (Note that the standard radial coordinate in physical space is $z^{\dagger}z$, while on our $\mathbf{R}^4$ it is given by $\sqrt{z^{\dagger}z}$!) We leave this somewhat sketchy, however, because our goal—to present a role of the Hopf fibration in the monopole context—has already been reached before.

We mention finally that the circle bundles corresponding to higher values of $|n|$ are so-called lens spaces. Roughly, they are the bundles of unit vectors of complex line bundles obtained by taking tensorial powers of the $n = \pm 1$ line bundle, as can be seen immediately from their transition functions. For connections on them, see [42,43] and Appendix A.

8. Hopf and the Dirac equation [9]

The two-level quantum systems of Section 2 may be considered, in particular, as describing the spin states of a non-relativistic spin-$\frac{1}{2}$ particle—i.e., one ignores its spatial degrees of freedom. If the latter is not done, one must write spinorial wave functions, subject to appropriate wave equations. In the relativistic regime, the relevant wave equation is the Dirac equation. We shall point out here that the Hopf fibration has significance also in this context. Most directly this comes about when one considers the analog of the Dirac equation in two-dimensional space-times in their Euclideanized and compactified versions, which is done either when studying field theories in two space-time dimensions for their own sake, or in (super)string theory. In this context, $\mathbf{S}_2 \cong \mathbf{CP}_1$ is just the simplest example of such a space-time or string world-sheet, more complicated ones being Riemann surfaces corresponding to algebraic curves [37]. The concept of spinor field, to be subjected to a Dirac equation, requires that of a *spin structure*, which is a 2:1 covering of the space-time's bundle of oriented orthonormal tangent frames (see, e.g., [44] for detailed exposition suitable in the present context). Now the assignment $z \mapsto \mathbf{Z}(z)$ used in Section 2 to graphically represent state vectors in the Bloch sphere picture of pure states shows that the Hopf bundle yields a 2:1 covering of the bundle of oriented orthonormal tangent frames of the 2-sphere, the structure group U(1) playing the role of the spin group covering the frame bundle's

[9] The idea to include this topic is due to A. Trautman.

structure group SO(2). Indeed one verifies by elementary calculation that $\mathbf{P}(z) = \Re\mathbf{Z}(z)$, $\mathbf{Q}(z) := \Im\mathbf{Z}(z)$ satisfy $\mathbf{P}^2 = \mathbf{Q}^2 = 1$, $\mathbf{P}\cdot\mathbf{Q} = 0$, $\mathbf{P}\times\mathbf{Q} = \mathbf{R}$. Thus the Hopf bundle is the spin bundle of the 2-sphere.[10]

The canonical connection on the Hopf bundle (cf. Appendix A) already played a role in Sections 2 and 7. Its associated covariant differential for sections of vector bundles associated to the Hopf bundle provides the necessary ingredient for the definition of the Dirac operator, since it is directly related to the Levi–Civitá connection on the 2-sphere, as already remarked in Section 2.

It is, however, not necessary to enter these esoteric two-dimensional worlds to see the Hopf fibration as a relevant structure in the realm of spinor fields: Trautman [44] has noted that the Dirac operator on 2-spheres appears when one tries to solve the Dirac equation in ordinary four-dimensional Minkowski space in presence of an external spherically symmetric electrostatic field by the method of separation of variables, using polar coordinates. Indeed, the operators j and k appearing in Section 71 and Section 53 of [6] and [14], respectively—just to quote two popular expositions—are essentially the Dirac operator on the 2-sphere: this is easy to see locally, and was shown to be true globally in [44]. As Trautman points out, there is a direct global analog for Dirac operators to the well-known splitting of the Laplacian on $\mathbf{R}^3$ into a radial operator and $r^{-2}\times$ (Laplacian on $\mathbf{S_2}$).[11] At least for the mathematically minded, this way of looking at the formal procedure in [6,14] will be more satisfactory; and it allows for interesting generalizations [44] and applications [45].

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Appendix A.[12]

A.1. Defining the Hopf fibration

In mathematics, the Hopf fibration is nowadays defined as follows ([4, p. 227]; [5, Section 16.14]). Take the complex vector space $\mathbf{C^2}$ of columns $z = (\,z_1 \quad z_2\,)^{\mathrm{T}}$, where

[10] Also, $z\otimes z \mapsto \boldsymbol{Z}\cdot \mathrm{d}\boldsymbol{R}_z$ turns out to give an explicit isomorphism between the tensorial square of the "tautological" line bundle (bundle of chiral spinors) associated to the Hopf bundle and the holomorphic cotangent bundle of $\mathbf{CP_1}$, mentioned in Appendix A; see [37] for the extension of this aspect to higher genus algebraic curves and its application in superstring theory.

[11] Even the otherwise rather complete monograph [15] misses this point.

[12] As a general reference for the concepts used below, we recommend the text [17].

$z_i \in \mathbf{C}$. Then the complex projective line[13] $\mathrm{P}(\mathbf{C^2}) = \mathbf{CP_1}$ obtains from $\mathbf{C^2} \setminus \{0\}$ by factoring out the equivalence relation "$z \sim w$ iff $w = \lambda z$ for some $\lambda \in \mathbf{C}^\times = \mathbf{C} \setminus \{0\}$". Since these equivalence classes bijectively correspond to the one-dimensional subspaces of $\mathbf{C^2}$, one can also define the points of the projective space to be just these subspaces. For its (complex) manifold structure (see [4, p. 75]; [5, Section 6.11]; [10, p. 134]). Instead, one can first restrict to the unit 3-sphere $\mathbf{S_3} \subset \mathbf{R^4} = \mathbf{C^2}$ given by the z having $z^\dagger z = 1$, and then factor out "$z \sim w$ iff $w = \lambda z$ for some λ from the unit circle $\mathbf{S_1} \cong \mathrm{U}(1) = \{\lambda \in \mathbf{C}, |\lambda| = 1\}$". Using the stereographic maps from $\mathbf{CP_1}$ to the 2-sphere $\mathbf{S_2}$

$$\zeta := \frac{z_2}{z_1} \mapsto \left(\Re\frac{2\zeta}{1+|\zeta|^2}, \Im\frac{2\zeta}{1+|\zeta|^2}, \frac{1-|\zeta|^2}{1+|\zeta|^2}\right),$$

wherever $z_1 \neq 0$ and

$$\zeta' := \frac{z_1}{z_2} \mapsto \left(\Re\frac{2\bar{\zeta}'}{1+|\zeta'|^2}, \Im\frac{2\bar{\zeta}'}{1+|\zeta'|^2}, \frac{1-|\zeta'|^2}{1+|\zeta'|^2}\right),$$

wherever $z_2 \neq 0$, one gets a smooth map π from $\mathbf{S_3}$ onto $\mathbf{S_2} \subset \mathbf{R^3}$ which we can also express as

$$z \mapsto (2\Re\bar{z}_1 z_2, 2\Im\bar{z}_1 z_2, |z_1|^2 - |z_2|^2) =: \boldsymbol{R}(z).$$

(ζ, ζ' are the inhomogeneous complex coordinates around 0 and ∞ in $\mathbf{CP_1}$). This is the Hopf map. π is a projection in the sense that $\mathbf{S_3}$ is a principal fiber bundle over the base space $\mathbf{S_2} \cong \mathbf{CP_1}$ with structure group U(1) ([5, Section 16.14]; [7, Section 4.4.4]; [9, p. 50]) which we will call the Hopf bundle or *Hopf fibration*.[14] Hopf actually started out with a version in real terms, considering instead of $\mathbf{C^2}$ its realification $\mathbf{R^4}$, i.e., columns $X = (X_1, \dots, X_4)^\mathrm{T}$, where $z_1 = X_1 + \mathrm{i}X_2$, $z_2 = X_3 + \mathrm{i}X_4$. The real counterpart of multiplying z by i is multiplying X by the matrix (a *complex structure* for $\mathbf{R^4}$)

$$J := \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

We also equip $\mathbf{C^2}$ with its standard Hermitian inner product $\langle z, w\rangle := z^\dagger w = \bar{z}_1 w_1 + \bar{z}_2 w_2$. In $\mathbf{R^4}$, with Y the real counterpart of w, this gives rise to the standard inner product $\langle\langle X, Y\rangle\rangle = \Re\langle z, w\rangle \equiv X^\mathrm{T} Y$ and an anti-symmetric non-degenerate bilinear form $\Phi(X, Y) := \Im\langle z, w\rangle \equiv \Re\langle \mathrm{i}z, w\rangle = \langle\langle JX, Y\rangle\rangle$. In terms of X, the unit sphere is $X^\mathrm{T} X = 1$, and the fibers of the Hopf bundle—the inverse images of the points of $\mathbf{S_2}$ under the Hopf map—turn out to be linked great circles on $\mathbf{S_3}$: the orbit of z under the U(1) action, $\alpha \mapsto z\,\mathrm{e}^{\mathrm{i}\alpha}$, is in real terms $\alpha \mapsto X \cos\alpha + JX \sin\alpha$, where X and JX are orthogonal unit vectors. We note that the tangent to the fiber at z is given by iz, thus in real terms at X by JX. The linkage

[13] We will write dimensions as subscripts except where they at the same time can be read as Cartesian powers; it should be clear from the context whether a real or a complex dimension is meant.

[14] Likewise, one has $\mathbf{C^2} \setminus \{0\}$ as a (holomorphic) principal fiber bundle over $\mathbf{CP_1}$ with structure group $\mathbf{C}^\times$.

can be visualized by stereographic projection of the 3-sphere to its equatorial 3-plane: one obtains one family of Villarceau circles on each torus of a system of nested coaxial concentric tori orthogonal to the unit 2-sphere in that 3-plane; see, e.g., [48] and works quoted there for detailed illustration, and [4, pp. 227–239], for more on linking. The linking of the fibers was used by Hopf to compute the homotopy group $\pi_3(\mathbf{S_2}) \cong \mathbf{Z}$, while nowadays this is done using the machinery of the exact homoptopy sequence of fiber bundles ([4, p. 209]; [5, Section 16.30, Ex. 5,6]).

It should be noted that this linking of the fibers is *not* in conflict with the local triviality of the bundle. Since it is needed in one of our applications, we make this explicit with respect to a bundle atlas consisting of two charts. Let $\mathbf{U}^+ = \mathbf{S_2} \setminus \{\text{south pole}\}$ and $\mathbf{U}^- = \mathbf{S_2} \setminus \{\text{north pole}\}$ be the open subsets where ζ and ζ' are well defined, respectively; together, they cover the 2-sphere. Then an explicit fiber-preserving diffeomorphism from $\pi^{-1}(\mathbf{U}^\pm)$ to the product bundles $\mathbf{U}^\pm \times \mathrm{U}(1)$—in which the fibers are not linked—is given by $(\, z_1 \quad z_2\,)^{\mathrm{T}} \mapsto (z_2/z_1, z_1/|z_1|)$ for $\mathbf{U}^+$ and $(\, z_1 \quad z_2\,)^{\mathrm{T}} \mapsto (z_1/z_2, z_2/|z_2|)$ for $\mathbf{U}^-$ with respective inverses $(\zeta, \mathrm{e}^{\mathrm{i}\alpha_1}) \mapsto (1+|\zeta|^2)^{-1/2}\, \mathrm{e}^{\mathrm{i}\alpha_1}(\, 1 \quad \zeta\,)^{\mathrm{T}}$ and $(\zeta', \mathrm{e}^{\mathrm{i}\alpha_2}) \mapsto (1+|\zeta'|^2)^{-1/2}\, \mathrm{e}^{\mathrm{i}\alpha_2}(\, \zeta' \quad 1\,)^{\mathrm{T}}$. The usual polar coordinates θ, ϕ centered at the north pole of the 2-sphere are related to the complex stereographic coordinate ζ by $\zeta = \tan(\theta/2)\, \mathrm{e}^{\mathrm{i}\phi}$ (projecting from the south pole to the equatorial plane). Taking $\alpha_1 = 0$ and $\alpha_2 = 0$ we get cross-sections s^+ and s^- of the Hopf bundle over $\mathbf{U}^+$ and $\mathbf{U}^-$ given by

$$s^+(\zeta) = (1+|\zeta|^2)^{-1/2} \begin{pmatrix} 1 \\ \zeta \end{pmatrix} = \begin{pmatrix} \cos\dfrac{\theta}{2} \\ \sin\dfrac{\theta}{2}\, \mathrm{e}^{\mathrm{i}\phi} \end{pmatrix}$$

and

$$s^-(\zeta') = (1+|\zeta'|^2)^{-1/2} \begin{pmatrix} \zeta' \\ 1 \end{pmatrix} = \begin{pmatrix} \cos\dfrac{\theta}{2}\, \mathrm{e}^{-\mathrm{i}\phi} \\ \sin\dfrac{\theta}{2} \end{pmatrix},$$

respectively, which on $\mathbf{U}^+ \cap \mathbf{U}^-$ are related by $s^+ = \mathrm{e}^{\mathrm{i}\phi} s^-$ with the *transition function* $\mathrm{e}^{\mathrm{i}\phi}$.[15] Instead of θ, ϕ, α_1 or θ, ϕ, α_2 one may also use $\theta, \phi, \psi := \alpha_1 + \alpha_2$ or $\theta, \alpha_1, \alpha_2$ as convenient coordinates on the bundle manifold $\mathbf{S_3}$:[16]

$$z_1 = \cos\frac{\theta}{2}\, \mathrm{e}^{\mathrm{i}\alpha_1} = \cos\frac{\theta}{2}\, \mathrm{e}^{\mathrm{i}((\psi-\phi)/2)}, \qquad z_2 = \sin\frac{\theta}{2}\, \mathrm{e}^{\mathrm{i}\alpha_2} = \sin\frac{\theta}{2}\, \mathrm{e}^{\mathrm{i}((\psi+\phi)/2)}.$$

A.2. Clifford parallels

Clifford parallelism in real projective 3-space with the elliptic version of non-Euclidean geometry may be described as the following instruction to “draw” “parallels” to a given

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15 Similarly, one has local trivializations of the bundle mentioned in the previous footnote using the holomorphic local sections $\zeta \mapsto (\, 1 \quad \zeta\,)^{\mathrm{T}}$ and $\zeta' \mapsto (\, \zeta' \quad 1\,)^{\mathrm{T}}$ with transition function ζ.

16 Note that the latter are not *bundle* coordinates. The former are, being essentially Euler angles when the 3-sphere is interpreted as the group SU(2) as below and thus as the covering group of the rotation group SO(3).

straight line ℓ. The geometry being specified by an "absolute" quadric of signature $++++$, thus having no real points but defining a real polarity, one complexifies ℓ and intersects it with the complexified quadric. This gives the complex-conjugate points P and $\bar{P}$. Through them there pass complex-conjugate generators g and $\bar{g}$, both taken from the same of the two families of straight lines (=generators) carried by the complexified absolute. (Due to the present signature, each family is invariant under complex conjugation.) Any complex line joining a point P' on g with its complex-conjugate $\bar{P}'$ on $\bar{g}$ is the complexification of a real line ℓ', which is then called a *Clifford parallel* of ℓ. Depending on which of the two families of generators is being used, one has a right and a left Clifford parallelism. One can show that Clifford parallels are skew and equidistant in the sense of elliptic geometry and may be defined without using complexification [3, p. 200 et seq.].

We now indicate the relation of the Hopf fibration to Clifford parallels. We stated above that when $\mathbf{C^2}$ with its Hermitian form $\langle\cdot,\cdot\rangle$ of signature $++$ is realified, i.e., regarded as $\mathbf{R^4}$, the latter comes with the quadratic form $\langle\langle\cdot,\cdot\rangle\rangle$ and the complex structure J. One now complexifies the $\mathbf{R^4}$ to become $\mathbf{C^4}$ and extends J and the bilinear form corresponding to $\langle\langle\cdot,\cdot\rangle\rangle$ linearly and bilinearly, respectively, denoting the extensions by the same symbols. From $J^2 = -\mathbf{1}$ it follows that iJ has eigenvalues ± 1, the corresponding two-dimensional eigenprojectors being $\frac{1}{2}(\mathbf{1} \pm \mathrm{i}J)$. The eigenspaces are complex-conjugates of each other, check to be totally null with respect to $\langle\langle\cdot,\cdot\rangle\rangle$ due to the relation $\langle\langle JX, JY\rangle\rangle = \langle\langle X, Y\rangle\rangle$, and to belong, due to signature $++++$, to the *same* family of 2-spaces contained in the null cone defined by $\langle\langle\cdot,\cdot\rangle\rangle$ (their Plücker tensors are both selfdual). Given now $z \in \mathbf{C^2}$, we have its realification X; we can form JX and the 2-space $\mathrm{span}(X, JX)$. Then the two null directions contained in the complexification of the latter are parallel to $X \pm \mathrm{i}JX$, thus belonging to the eigenspaces just introduced. Projectivizing everything we see that the projectivized spaces $\mathrm{span}(X, JX)$ are indeed Clifford-parallel straight lines of $\mathbf{RP_3} = P(\mathbf{R^4})$ with the projectivized $\langle\langle\cdot,\cdot\rangle\rangle$ for its absolute. (Note that one gets a system of Clifford parallels for any J sharing the properties just written. Also remember that the process of projectivization identifies anti-podal points on $\mathbf{S_3}$.)

A.3. Symmetry group

The Hopf fibration has a symmetry group: the action of the unitary group U(2) on $\mathbf{C^2}$ leaves the 3-sphere invariant and carries fibers into fibers, as it commutes with the U(1) action; it thus descends to an action on the 2-sphere by ordinary rotations. This gives nothing but the well-known covering homomorphism $\mathrm{SU}(2) \to \mathrm{SO}(3)$, as will transpire from the quantum mechanical interpretation in Section 2. The difference between U(2) and SU(2) is ineffective on the 2-sphere due to the factoring out of the structure group U(1).

A.4. Group theoretic definition

It will be useful for our purposes to give a slightly modified description of the fibering of the 3-sphere. To the columns z we attach their unitary perpendiculars to form the 2×2 matrices $\mathrm{U}(z) = \begin{pmatrix} z_1 & -\bar{z}_2 \\ z_2 & \bar{z}_1 \end{pmatrix}$: if $z^\dagger z = 1$, they make up the group $\mathrm{SU}(2) \cong \mathbf{S_3}$. (If z is left arbitrary, one obtains a matrix ring isomorphic to the skew field of quaternions.) The U(1) action on z can be replaced here by multiplying on the right by $\mathrm{diag}(\lambda, \lambda^{-1}) \in \mathrm{SU}(2)$, so that the Hopf fibers are just left cosets of a U(1) subgroup in SU(2) embedded as indicated, and we now have the 2-sphere as the coset space SU(2)/U(1). The above-mentioned U(2) symmetry of the fibration corresponds to left multiplication by SU(2) matrices in addition to right multiplication by the U(1) subgroup elements.[17] One can also consider right cosets of that U(1) subgroup, resulting in the *anti-Hopf fibration*—its stereographic projection gives the second family of Villarceau circles on the tori mentioned above, with the opposite sense of linking. Embedding U(1) into SU(2) by $\lambda = \mathrm{e}^{\mathrm{i}\alpha} \mapsto \exp(\mathrm{i}\alpha n^k \sigma_k)$ (where the σ_k are the Pauli matrices; these subgroups are all conjugate to the one above) one gets all other systems of Clifford parallels, one for each pair $\pm\boldsymbol{n}$ of real unit vectors.

The standard metric $\mathrm{d}s^2 = \mathrm{d}z_1\,\mathrm{d}\bar{z}_1 + \mathrm{d}z_2\,\mathrm{d}\bar{z}_2$ of $\mathbf{S_3}$ can also be written as $\mathrm{d}s^2 = \det \mathrm{d}U = \det \Omega$, where Ω is the left or right invariant Maurer–Cartan matrix on SU(2), $U^{-1}\,\mathrm{d}U$ or $(\mathrm{d}U)U^{-1}$, and where the products of differentials are symmetric tensor products as usual. Ω takes values in the matrix Lie algebra of SU(2) which consists of traceless anti-Hermitian matrices. If it is decomposed as $\Omega = (-\mathrm{i}/2)\omega^k\sigma_k$, the ω^k constitute a basis of left- or right-invariant 1-forms on the group manifold. They are used in Section 4. In terms of them, one gets $\mathrm{d}s^2 = \frac{1}{4}\delta_{jk}\omega^j\omega^k$, so that the $\omega^k/2$ form an orthonormal (co)basis and the Riemannian volume form is given by $-\frac{1}{8}\omega^1 \wedge \omega^2 \wedge \omega^3$. (The sign has been chosen so as to conform with the standard orientation of the 3-sphere as embedded into $\mathbf{R^4}$ as we have it here.) For the left-invariant forms one has $d \wedge \Omega = -\Omega \wedge \Omega$, implying the Maurer–Cartan relations $d \wedge \omega^\ell = -\frac{1}{2}\epsilon_{\ell jk}\omega^j \wedge \omega^k$. In particular, for $\omega^3 = 2\mathrm{i}(\bar{z}_1\,\mathrm{d}z_1 + \bar{z}_2\,\mathrm{d}z_2) = 2\mathrm{i}z^\dagger\,\mathrm{d}z$ one has $\mathrm{d}\omega^3 = -\omega^1 \wedge \omega^2$, so that the volume element also equals $\frac{1}{8}\omega^3 \wedge d \wedge \omega^3$.

The exterior 2-form globally defined on $\mathbf{R^3}$ by $\boldsymbol{R} \cdot \mathrm{d}\boldsymbol{R} \times \mathrm{d}\boldsymbol{R}$ restricts to the usual surface element Σ on the unit 2-sphere with its standard orientation (locally, $\Sigma = \sin\theta\,\mathrm{d}\theta \wedge \mathrm{d}\phi$ in polar coordinates). Σ is trivially closed but not exact, as its integral over $\mathbf{S_2}$ is $4\pi \neq 0$. When pulled back to $\mathbf{S_3}$ by the Hopf map $z \mapsto \boldsymbol{R}(z)$, some computation (using $z^\dagger z = 1$) yields the 2-form $2\mathrm{i}(\mathrm{d}z_1 \wedge \mathrm{d}\bar{z}_1 + \mathrm{d}z_2 \wedge \mathrm{d}z_2) \equiv \mathrm{d} \wedge (-\omega^3)$, which is manifestly exact on $\mathbf{S_3}$. Generally, for a smooth map $f : \mathbf{S_3} \to \mathbf{S_2}$, the pullback $f^*((4\pi)^{-1}\Sigma) = d \wedge \alpha$ is exact since it is closed as the pullback of a closed form and since all closed 2-forms on $\mathbf{S_3}$ are exact. The integral $\int_{\mathbf{S_3}} \alpha \wedge d \wedge \alpha$ is called the Hopf invariant of f; it can be shown [4] to be an integer equal to the linking number of the inverse images of any two points in $\mathbf{S_2}$ under f. For the Hopf map, this number becomes $(4\pi)^{-2}\int_{\mathbf{S_3}} \omega^3 \wedge d \wedge \omega^3 = (2\pi^2)^{-1} \cdot \mathrm{volume}(\mathbf{S_3}) = 1$.

Note in this context that for any 1-form α the (non)vanishing of the integrand $\alpha \wedge d \wedge \alpha$ is the Frobenius criterion[18] for the local (non)existence of 2-surfaces on which α restricts to zero or which are orthogonal to the flow lines of the vector field metric-related to α. In case of non-existence, the vector field is said to be *twisting*.

Locally, using the parametrization by ψ, θ, ϕ, the left-invariant 1-form $z^\dagger\,\mathrm{d}z$ equals $(\mathrm{i}/2)(\mathrm{d}\psi - \cos\theta\,\mathrm{d}\theta)$, and the Hopf map is $(\psi, \theta, \phi) \to (\theta, \phi)$ (so that locally it is

[17] So the symmetry group seems to be $\mathrm{SU}(2) \times \mathrm{U}(1)$, but the action has a kernel $\mathbf{Z}_2$ consisting of $(\mathbf{1}, \mathbf{1})$ and $(-\mathbf{1}, -\mathbf{1})$; the quotient is isomorphic to U(2).

[18] For the interesting history of this criterion and its generalization, see [39].

immediate that the surface element of the 2-sphere pulls back to the exterior derivative of $-\omega^3$, but notice the dangers of sloppy notation!). It will be met again below.

A.5. Associated line bundles: lens spaces

We will encounter complex line bundles associated to the Hopf bundle, the basic one being the *tautological line bundle over* $\mathbf{CP_1}$.[19] This is $\mathbf{C^2}$ with its origin "blown up", i.e., the subset of the trivial rank 2 vector bundle $\mathbf{CP}_1 \times \mathbf{C^2}$ consisting of those pairs (P, z), where z belongs to the one-dimensional subspace of $\mathbf{C^2}$ determined by the equivalence class $P \in \mathbf{CP_1}$. The others will be tensorial powers of it or of its dual ("negative" tensorial powers; the zeroth power is defined to be the trivial line bundle $\mathbf{CP_1} \times \mathbf{C}$). Equivalently, these are associated to the Hopf bundle in the sense of forming $(\mathbf{S_3} \times \mathbf{C})/\mathrm{U}(1)$, where $\mathrm{U}(1)$ acts on the "standard fiber" $\mathbf{C}$ by some of its one-dimensional representations $\lambda \mapsto \lambda^n$, $n \in \mathbf{Z}$ ([5, Section 16.14.7]; [7, Section 4.4.5]; [9, p. 54]).

The circle bundles formed by their unit vectors—the Hopf bundle is just the bundle of unit vectors, in the sense of $\langle \cdot, \cdot \rangle$, of the tautological one—are so-called *lens spaces* $\mathrm{L}(3, n)$ (employing the notation of [4]—we warn the reader that there are other notations for them, depending on the generalizations envisaged). A related construction of $\mathrm{L}(3, n)$ (cf. [42]) is by first considering the circle bundle $\mathbf{S_{2n+1}} \to \mathbf{CP_n}$—defined in complete analogy to our $\mathbf{S_3} \to \mathbf{CP_1}$—and then considering the circle bundle induced [9, p. 60] from there over $\mathbf{CP_1}$ under the Veronese[20] embedding $\mathbf{CP_1} \to \mathbf{CP_n}$. The most direct definition [4, p. 243] consists in quotienting $\mathbf{S_3}$ by the (free) action of the subgroup $\mathbf{Z}_n \subset \mathrm{U}(1)$ generated by $\exp(2\pi \mathrm{i}/n)$: thus $\mathbf{S_3} \to \mathbf{S_3}/\mathbf{Z}_n = \mathrm{L}(3, n)$ is an n-fold covering, and, writing $[z]$ for z mod $\mathbf{Z_n}$, we have the $\mathrm{U}(1)$ action $[z]\lambda := [z\lambda^{1/n}]$ on $\mathrm{L}(3, n)$ which makes it into a principal $\mathrm{U}(1)$ bundle over the 2-sphere, homomorphic [9, p. 53] to the Hopf bundle via the covering map and the homomorphism $\mathrm{U}(1) \to \mathrm{U}(1)$ given by $\lambda \mapsto \lambda^n$. The standard metric $\mathrm{d}z^\dagger\, \mathrm{d}z$ on the 3-sphere descends to a metric on $\mathrm{L}(3, n)$ which is thus one of the positive curvature Clifford–Klein space forms;[21] notice that the identification of points in the transition to $\mathrm{L}(3, n)$ reduces the *global* isometry group of the 3-sphere to $\mathrm{SU}(2) \times \mathrm{U}(1)/\{\mathbf{1}, -\mathbf{1}\}$ for $\mathrm{L}(3, n)$, if $n > 2$. The lens space $\mathrm{L}(3, 2)$ is just real projective 3-space as we used it above for Clifford parallelism, or, using the group description of the Hopf fibration, $\mathrm{L}(3, 2) = \mathrm{SU}(2)/\mathbf{Z_2} \cong \mathrm{SO}(3)$; as a bundle over the 2-sphere, it is isomorphic to the latter's bundle of oriented orthonormal tangent frames. The Hopf bundle is thus the *spin bundle* of the 2-sphere, so that the tautological bundle is also the bundle of the 2-sphere's semispinors of one chirality. (The other chirality corresponds to the bundle dual to the tautological one.)

We shall not need the holomorphic aspects of the line bundles above; but let it be mentioned that these bundles are similarly associated to the holomorphic principal bundle that came up when we introduced the Hopf fibration, and thus are holomorphic; the square of

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[19] Apart from the blowing-up of the origin of $\mathbf{C^2}$, the fiber over a point of $\mathbf{CP_1}$ is the one-dimensional subspace of $\mathbf{C^2}$ which a point of $\mathbf{CP_1}$ is by definition—hence the name. Other names to be encountered are natural, canonical, universal (cf. [4, p. 268]; [5, Section 16.16, ex. 1, 2]; [7, p. 13]; [10, p. 306]).

[20] Suitably identifying $\mathbf{C_{n+1}}$ with the nth symmetric tensorial power of $\mathbf{C^2}$, one gets an isometric map $\mathbf{C^2} \to \mathbf{C^{n+1}}$ by assigning to $z \in \mathbf{C^2}$ its nth tensorial power $z^{\otimes n}$; this descends to an embedding of the projectivized spaces.

[21] These spaces are sometimes considered by cosmologists [24].

the tautological bundle is holomorphically equivalent to the holomorphic cotangent bundle of $\mathbf{CP_1}$; its positive powers admit only the zero section for holomorphic sections, while if $n \leq 0$ its nth powers possess $(-n+1)$-dimensional spaces of holomorphic sections. (This is best seen by series expansion using the local trivializations mentioned.)

A.6. Connections

A *connection* in a principal fiber bundle is specified geometrically by choosing, in each tangent space to the bundle manifold, a complement to the tangent space of the fiber in such a way that the distribution of these complements (the *horizontal subspaces*) is smooth and invariant under the action of the structure group ([5, Section 20.2]; [9, p. 63]; [41]). Equivalently, one can use the *connection form* ω, defined in those references, which takes values in the Lie algebra of the structure group and annihilates the horizontal subspaces. In the Hopf bundle, a particular and distinguished connection is obtained by taking the horizontal subspaces to be orthogonal to the fibers; it is sometimes called the canonical connection, and it obviously shares the full symmetry possessed by the fibration. (For its definition in terms of the group theory description of the bundle and in terms of the holomorphic aspects, see [9, p. 103] and [10, p. 178], respectively.) *Parallel transport* of points in the bundle space over a given curve in the base space means going along a curve in the bundle space that projects to the given one in the base and is tangent to the horizontal subspace at each of its points. Thus for the canonical connection of the Hopf bundle these curves are orthogonal to the Hopf circles—so their tangents $\dot{X}$ at position X must satisfy $\langle\langle X, \dot{X}\rangle\rangle = 0$ to be tangent to the 3-sphere and $\langle\langle JX, \dot{X}\rangle\rangle = 0$ to be orthogonal to the fiber. This gives $\langle z, \dot{z}\rangle = 0$ as the equation for parallel transport. The connection form for the canonical connection is $\langle z, \mathrm{d}z\rangle \equiv z^{\dagger}\mathrm{d}z$; it is obviously also invariant under the full symmetry group of the bundle.

Since the distribution of horizontal subspaces for a connection on the Hopf bundle is to be invariant under the action of the structure group—and therefore under its $\mathbf{Z}_n$ subgroups—one can use the covering projection to the lens space $\mathrm{L}(3, n)$ to define a connection there; its connection form is such that its pullback to the Hopf bundle via the covering projection equals n times the connection form one started with, and the same also holds for the *curvature form* [9, p. 79]. This implies, in particular, that the Chern class [10, p. 305] of $\mathrm{L}(3, n)$ or the associated line bundle is n times the one for Hopf. In particular, from the canonical connection one gets a connection on each $\mathrm{L}(3, n)$ whose horizontal subspaces are again orthogonal to the fibers in the sense of the metric mentioned above, again possessing the same symmetry group. One can also obtain it by pulling back [9, p. 82], via the Veronese embedding [42], the canonical connection form on the U(1) bundle $\mathbf{S_{2n+1}} \to \mathbf{CP^n}$, which is similarly defined. The connection obtained in $\mathrm{L}(3, 2)$ from the canonical connection on the Hopf bundle is nothing but the Levi–Cività connection on the 2-sphere, as is clear from its symmetry; conversely, the canonical connection on the Hopf bundle is the *spin connection* for Levi–Cività.

A.7. CR-structure

The horizontal subspaces of the canonical connection, orthogonal to the Hopf fibers, enjoy another property that has physical relevance mentioned in Section 4. They are the maximal subspaces of the tangent spaces to the 3-sphere that are *complex* subspaces of the embedding $\mathbf{C}^2$, since being $\langle , \rangle$-orthogonal to z is a complex-linear condition. This presents the 3-sphere as a *CR-manifold*; but we refer the reader to [8,46] for any further discussion of this aspect.

A.8. Covariant differentiation: local gauges

Along with a connection in a principal bundle P comes *covariant differentiation* ∇ of (smooth) sections ψ of associated vector bundles E ([5, Section 20.3]; [9, p. 113]; [41]). There are two equivalent ways of describing it invariantly and globally, one using parallel transport in associated bundles and one using the bijective correspondence that exists between such sections and functions $\tilde{\psi}$ on the principal bundle with values in the standard fiber F of the associated one, subject to a certain equivariance condition. (We make use of this correspondence in Section 6.[22]) One derives from this (cf. [41]) a description using local trivializations (*local gauge choices*, in physics terminology), and that is the way we need for comparison with the monopole situation.

Let ω be a connection form on the principal bundle space P, and ψ cross-section of the associated vector bundle $E \to M$, where the structure group G acts on the standard fiber F of the latter by the representation σ (and its Lie algebra by the representation $\dot{\sigma}$); let $\mathbf{U} \subset M$ be an open subset of the base space M and $s : \mathbf{U} \to P$ be a local section. Then ψ is locally given by the F-valued function $\psi_s := \tilde{\psi} \circ s$ on $\mathbf{U}$ and its covariant differential by the F-valued 1-form $\mathrm{d}\psi_s + \Gamma\psi_s$ on $\mathbf{U}$, where $\Gamma := \dot{\sigma}(s^*\omega)$. In Section 7, we are interested in $\mathrm{G} = \mathrm{U}(1)$ and complex line bundles—thus in unitary representations σ on $F = \mathbf{C}$; they are given by the characters $\sigma : \mathrm{e}^{\mathrm{i}\alpha} \mapsto \mathrm{e}^{\mathrm{i}n\alpha}$, $n \in \mathbf{Z}$. The forms ω, $s^*\omega$ and Γ are then purely imaginary, like $\omega = z^\dagger \mathrm{d}z$ on $\mathbf{S_3}$. The local versions $\partial_k + \Gamma_k$ of the covariant derivative, with $\Gamma_k = (\mathrm{i}e/c)A_k$, are used in Section 7 for writing the Hamiltonian of an electrically charged particle in a magnetic field.

The description of principal bundles in terms of local trivializations ([5, Section 16.13]; [9, p. 51]) is particularly simple in the case of the Hopf bundle (and the lens spaces derived from it), since one needs only a covering of the basis 2-sphere by the two open subsets $\mathbf{U}^\pm$, containing the upper $(+)$ and lower $(-)$ closed hemispheres $\mathbf{H}^\pm$: over these contractible subsets, the bundles are trivial, so sections exist, and one has only to write down the one single transition function ("clutching function") between them on $\mathbf{U}^+ \cap \mathbf{U}^-$. For Hopf and anti-Hopf, we did this explicitly before, and we get it immediately for the lens spaces—as we defined them—to be $\mathrm{e}^{\mathrm{i}n\phi}$. (Bott and Tu [4, p. 301] teach us that this already exhausts all circle bundles and associated complex line bundles over the 2-sphere, up to isomorphism, since n fixes the homotopy class of the clutching function.)

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